by the solution of a completely integrable system of partial differential equations of the first order a family of parallel nets of a particular type are obtained, each of which determines a T transform N_1, which also is a permanent net. All of these transformations are now shown to give a solution of Problem A.

In the transformations just referred to we did not consider permanent nets for which the curves in one family are represented on the Gauss sphere by one system of the imaginary generators. Drach[4] solved the problem of the deformation of nets of this kind. We show how in two ways these nets can be transformed into nets of the same kind as a solution of Problem A.

The third type of permanent nets are those whose two families of curves are represented on the sphere by its isotropic generators. These curves are the minimal lines on a minimal surface. There are no transformations of nets of this kind into similar nets furnishing a solution of Problem A.

[1] Eisenhart, *Trans. Amer. Math. Soc., New York,* **18,** 1917, (97-124).
[2] Peterson, *Ueber Curven und Flachen,* Moskau and Leipzig, 1868, (106).
[3] Eisenhart, *Rend. Circ. Mat., Palermo,* **39,** 1915, (153-176).
[4] Drach, *Ann. Fac. Sci. Toulouse,* (Ser. 2), **10,** 1908, (125-164).

ON BILINEAR AND N-LINEAR FUNCTIONALS

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It has been proved by Riesz[1,2] that if a linear functional $A(f(s))$ is continuous with zeroth order, there is a unique regular function $\alpha\ (s)$ which satisfies the equation

$$A\,(f)\ = \int f\,(s)\,d\alpha\,(s),$$

and the variation of α is the least upper bound of the expression $|A(f)|/mf$, where mf is the maximum of $|f(s)|$. From this theorem Fréchet][3] has proved that if $U\,(f\,(s), g\,(t))$ is bilinear, that is linear in each argument, there is a function $u\ (s, t)$ which is regular in t and satisfies the equation

$$U\,(f, g) = \int\int f\,(s)\ g\,(t)\ d_2u\ (s,\,t), \tag{1}$$

and by modifying the definition of the variation of a function of two variables, he has proved that the variation of $u\,(s,t)$ is the least upper bound of $|\ U\,(f, g)\ |/mfmg$.

In the present note equation (1) is derived by a different method, and the function $u(s, t)$ is proved to be regular in both arguments when determined by this method, and unique if regular. The theorem can then be extended by mathematical induction to n-linear functionals.

The variation V_2u of $u(s, t)$ is defined by Fréchet as the least upper bound of the expression

$$\sum_{i,j} e_i e_j \, \Delta_{ij} u(s, t),$$

where e_i and e_j are taken equal to plus or minus 1 in such a way as to make the summation as large as possible, and

$$\Delta_{ij} u(s, t) = u(s_{i+1}, t_{j+1}) - u(s_i, t_{j+1}) - u(s_{j+1}, t_j) + u(s_i, t_j).$$

The double Stieltjes integrals considered here will all be of the form

$$\int_T f(s)\, g(t)\, d_2 u(s, t) = \text{limit} \sum_{i,j} f(s_i')\, g(t_j')\, \Delta_{ij} u(s, t),$$

where the region T is defined by the inequalities

$$T: \; a \leqq s \leqq a'; \; b \leqq t \leqq b',$$

and s_i' and t_j' are in the intervals (s_i, s_{i+1}) and (t_j, t_{j+1}) respectively. Such an integral always exists if f and g are continuous and V_2u is finite, and it must satisfy the inequality

$$\left| \int_T f(s)\, g(t)\, d_2 u(s, t) \right| \leqq mf mg V_2 u. \text{[3]} \qquad (2)$$

A function $u(s,t)$ will be called regular here if V_2u is finite, $u(a,t) = u(s, b) = 0$, and $u(s, t) = u(s+0, t) = u(s, t+0)$ excepting on the boundary of T. This makes some of the work simpler than to assume with Fréchet that $2u(s, t) = u(s - 0, t) + u(s + 0, t)$. If $u(s, t)$ is regular, its variation in one variable, when the other is constant, cannot be greater than V_2u. The double integral can then be expressed as an iterated integral by the equation[3]

$$\int_T f(s)\, g(t)\, d_2 u(s, t) = \int_a^{a'} f(s)\, d_s \int_b^{b'} g(t)\, d_t u(s, t).$$

The functional

$$v(g(t); s) = \int_b^{b'} g(t)\, d_t u(s, t)$$

can be proved regular in s by proving that its variation cannot be greater than mgV_2u, and then proving that $v(g; s + \epsilon)$ approaches $v(g; s)$ when ϵ

approaches zero. The details of the proof of this will not be given here.

If two regular functions $u(s, t)$ and $u'(s, t)$ satisfy the equation

$$\int_T f(s)\, g(t)\, d_2 u(s, t) = \int_T f(s)\, g(t)\, d_2 u'(s, t) \tag{3}$$

identically in f and g, they will be proved to be identical. Since the similar theorem has been proved for single Stieltjes integrals, if u and u' are unequal at a point (s', t') there must be a continuous function $g(t)$ such that $v(g; s') \neq v'(g; s')$, where v' is the functional analogous to $v(g; s)$. Then since v and v' are regular in s there must be a continuous function $f(s)$ for which the two members of equation (3) are unequal. This proves that if a functional $U(f, g)$ satisfies the equation (1) where $u(s, t)$ is regular and independent of f and g, the function $u(s, t)$ must be unique.

As Riesz has proved,[2] the field of functions for which a linear functional is defined can be extended to any function which is the limit of a sequence of continuous functions which satisfy the inequalities

$$f_1 \geqq f_2 \geqq f_3 \geqq \ldots, \tag{4}$$

and any linear combination of such functions. Thus if $U(f, g)$ is bilinear, one or both of the functions f and g may be discontinuous if it is the limit of such a sequence. A function $f(s, s')$ will be defined by the equations

$$\begin{aligned} &f(s, a) = 0, \\ &f(s, s') = 1, \qquad (a \leqq s \leqq s';\ s' > a), \\ &f(s, s') = 0, \qquad (s' < s \leqq a'), \end{aligned}$$

and a function $g(t, t')$ by the analogous equations. If f is considered a function of s it is approached by a sequence such as (4), as Riesz has shown,[2] and $g(t, t')$ has the same property. The function $u(s, t)$ will then be defined by the equation

$$u(s', t') = U(f(s, s'), g(t, t')).$$

This function vanishes for $s' = a$, or $t' = b'$, by definition. Since U is bilinear the expression $U(f, g)/mfmg$ is bounded[3] and its least upper bound will be called M, a constant independent of f and g. The variation of u is defined as the upper bound of the expression

$$\sum_{i,j} e_i e_j\, \Delta_{ij} u(s', t') = U\Big(\sum_i e_i\,(f(s, s'_{i+1}) - f(s, s'_i)) \sum_j e_j\,(g(t, t'_{j+1}) - g(t, t'_j))\Big),$$

and since from definition

$$\sum e_i \left(f(s, s'_{i-1}) - f(s, s'_i)\right) \leqq 1$$

the right member of the last equation cannot be greater than M. That is

$$V_2 u(s, t) \leqq M. \tag{5}$$

It can be proved by the method Riesz uses in showing that a linear functional is defined uniquely for the limit of a sequence such as (4), that the function $u(s, t)$ just defined is the limit of $u(s+\epsilon, t)$ when $\epsilon = 0$. Thus $u(s, t)$ is regular.

It will now be proved that

$$U(f, g) = \int_T f(s)\, g(t)\, d_2 u(s, t), \tag{6}$$

where $f(s)$ and $g(t)$ are arbitrary continuous functions. The function $f'(s)$ will be defined by the equation

$$f'(s) = \sum_i f(s'_i) \left(f(s, s'_{i+1}) - f(s, s'_i)\right),$$

and $g'(t)$ in the analogous way. Then the equation

$$U(f', g') = \sum_{i,j} f(s'_i)\, g(t'_j)\, \Delta_{ij} u(s', t') \tag{7}$$

will be satisfied. It follows from definition that $f'(s) = f(s'_i)$ in the region $s'_i < s \leqq s'_{i+1}$, and similarly for g' and g. Since U is linear in each argument and f and g are continuous, $U(f'_i, g')$ approaches $U(f, g)$ when the length of the greatest of the intervals approaches zero, and the right member of equation (7) approaches the Stieltjes integral in equation (6). Inequalities (2) and (5) imply that $V_2 u$ is equal to M. This completes the proof that when $U(f, g)$ is bilinear there is a unique function $u(s, t)$ which is regular and satisfies equation (6), and the variation of u is the least upper bound of $|U(f, g)|/mfmg$.

If this property is assumed for functionals linear in each of $n-1$ arguments, the proof just outlined can be modified to make it prove that functionals linear in n arguments have the same property. Thus the theorem holds for n-linear functionals.

This can be used to extend a theorem of Fréchet's[3] about functionals of the second order to those of the nth order. A functional of the nth order is defined as one that is continuous and satisfies the equation

$$U(f_1+f_2+\ldots+f_{n+1})-\sum_{r=1}^{n+1}U(f_1+\ldots+f_{r-1}+f_{r+1}+\ldots+f_{n+1})+\sum_{r=1}^{n}\sum_{s=r+1}^{n+1}U(f_1+\ldots+f_{r-1}+f_{r+1}+\ldots+f_{s-1}+f_{s+1}+\ldots+f_{n+1})\ldots+(-1)^{n+1}U(0)=0$$

identically. If $U(f)$ is also homogeneous, the functional defined by the equation

$$W(f_1,f_2,\ldots,f_n)=U(f_1+\ldots+f_n)-\sum_{r=1}^{n}U(f_1+\ldots+f_{r-1}+f_{r+1}+\ldots+f_n)+\ldots+(-1)^n U(0)$$

is easily proved to be linear in each of its arguments. If $f_i=f_2=\ldots=f$, it follows from definition and the condition of homogeneity, that

$$W(f,f,\ldots,f)=K_n U(f),$$

where

$$K_n=\sum_{r=0}^{n-1}(-1)^r\frac{n(n-1)\ldots(n-r+1)}{r!}(n-r)^n.$$

But this expression can be proved equal to $n!$ which cannot vanish for any positive value of n, and since the n-linear functional W can be expressed as a multiple Stieltjes integral the homogeneous functional $U(f)$ of order n can be put in the same form.

[1] Riesz, *Ann. Sci. Ec. norm., Paris,* (Ser. 3), **28,** 1911, (36–43).

[2] *Ibid.,* **31,** 1914, (9–14).

[3] Fréchet, *New York, Trans. Amer. Math. Soc.,* **16,** 1915, (215–234).

THE CRYSTAL STRUCTURE OF CHALCOPYRITE DETERMINED BY X RAYS

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Introduction.—This investigation of the atomic structure of crystals of chalcopyrite ($CuFeS_2$) was undertaken, as no study of a complex sulfide by the method of X-rays had previously been carried out. Moreover, comparatively few crystals of the tetragonal system, in which chalcopyrite crystallizes, have been examined; the only ones being certain oxides of the formula MO_2 studied by Vegard[1] and by Williams.[2] Yet the determination of the structure of crystals belong-